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Some Properties of Chebyshev Approximation in a Subspace of R^n

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Let K be a subspace of R^n and let $K^\perp$ be the orthogonal complement of K. Rockafellar has shown that certain properties of K may be characterized by considering the possible patterns of signs of the nonzero components of vectors of K and of $K^\perp$. Such considerations are shown to lead to the standard characterization theorem for discrete linear Chebyshev approximation as well as to several results on uniqueness of solutions. A method is given for testing uniqueness of a given solution. A special case related to graph theory is discussed and combinatorial methods are given for solving and testing for uniqueness.

1. INTRODUCTION

Let K be a subspace of R^n and let $K^\perp$ be the orthogonal complement of K. Rockafeller has shown in [7] that certain properties of K may be characterized by considering the possible patterns of signs of the nonzero components of vectors of K and of $K^\perp$. We show here that such considerations lead to the standard characterization theorem for discrete linear Chebyshev approximation as well as to several results on uniqueness of solutions. We give a method for testing uniqueness of a known solution. Also, we discuss a Chebyshev approximation problem concerning a real-valued function on a subset of $W \times W$, where W is a finite set. By reference to the concept of tension in a graph, we show the problem to be a special case of Chebyshev approximation in a linear subspace. We describe combinatorial methods for solving this special problem and testing a given solution for uniqueness.

2. PRELIMINARIES

We view a vector $x = (x_1, ..., x_n)$ as a real-valued function on a set $E = \{e_1, ..., e_n\}$. The *support* of x, denoted $S(x)$, is the subset of E on which

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the function is nonzero. The number of elements in $S(x)$ is denoted $|S(x)|$. We say that x *conforms to* y *on* $M \subseteq E$ if $[e_i \in M$ and $x_i \neq 0] \Rightarrow x_i y_i > 0$. Thus, that statement implies that $M \cap S(x) \subseteq M \cap S(y)$. The statement "$x$ conforms to y" means that x conforms to y on E. (If x conforms to y then $S(x) \subseteq S(y)$.)

If x and y belong to R^n, let $x \cdot y$ denote $x_1 y_1 + \cdots + x_n y_n$. The following is a (slightly weaker) version of a result given by Rockafeller [6].

THEOREM 2.1. *Let K be a subspace of R^n. Let $K^\perp$ be the orthogonal complement of K. Let $I_1, \ldots, I_n$ be arbitrary real intervals (not necessarily closed, open, or bounded, possibly a single point). Then, one and only one of the following alternatives holds:*

(a) *There exists a vector $u \in K$ such that $u_i \in I_i$ for $i = 1, \ldots, n$.*

(b) *There exists a vector of $K^\perp$, v, such that $v \cdot w > 0$ for all $w \in R^n$ satisfying $w_i \in I_i$ for $i = 1, \ldots, n$.*

A *spanning vector* of K is a vector $x \in K$ such that for every nonzero $y \in K^\perp$, $S(x) \cap S(y) \neq \varnothing$. Suppose the q columns of matrix A form a basis for $K^\perp$. Let $\{i_1, \ldots, i_t\}$ be the indices of a subset of the rows of A and consider the submatrix consisting of these rows. The q columns of the submatrix are linearly dependent if and only if the rank of the submatrix is less than q and hence, there exists nonzero $y \in K^\perp$ such that $y_{i_j} = 0$ for $j = 1, \ldots, t$ if and only if this rank is less than q. Thus, $x \in K$ is a spanning vector of K if and only if the rows of A whose indices correspond to the indices of the nonzero components of x form a submatrix of rank q.

The following will be used in the next section:

THEOREM 2.2. *Let K be a subspace of R^n. Then, K possesses a spanning vector if and only if no vector of $K^\perp$ has exactly one nonzero component.*

Proof. Suppose that no vector of $K^\perp$ has exactly one nonzero component. Let x be a vector of K whose support has maximum cardinality. Suppose that $x_j = 0$. There must exist $z \in K$ such that $z_j \neq 0$ or there exists $w \in K^\perp$ with $S(w) = \{e_j\}$. Let

$$\lambda = (1/2) \min_{x_i \neq 0} |x_i| / \max_i |z_i|.$$

Then, $|S(x + \lambda z)| > |S(x)|$, contradicting the assumption about x. Therefore $x_j \neq 0$ and $S(x) = E$. Hence, x is a spanning vector of K. Conversely, if $y \in K^\perp$ has exactly one nonzero component, orthogonality implies that $S(y) \cap S(x) = \varnothing$ for every $x \in K$. Therefore, K has no spanning vector. ∎

3. Characterization and Uniqueness

The problem we study is the following: Let K be a subspace of R^n and let $b \in R^n$. (We are primarily interested in the case where $b \notin K$.) We wish to find $u^* \in K$ such that

$$\| u^* - b \|_\infty = \max_i | u_i^* - b_i | \leqslant \| u - b \|_\infty \qquad \forall u \in K. \tag{3.1}$$

A proof of the existence of u^* satisfying (3.1) appears in [5]. The following restates the characterization theorem given in [4].

THEOREM 3.1. *Let K be a subspace of R^n and let $b \in R^n$, $b \notin K$. Let $u^* \in R^n$, $r^* = u^* - b$, $T^* = \{e_j : | r_j^* | = \| r^* \|_\infty\}$. Then, u^* satisfies* (3.1) *if and only if there exists nonzero $v \in K^\perp$ such that $S(v) \subseteq T^*$ and v conforms to r^*.*

Proof. Let u^* be given and let there exist nonzero $v \in K^\perp$ such that $S(v) \subseteq T^*$ and v conforms to r^*. Suppose that u^* does not satisfy (3.1). Then, there exists $\bar{u} \in K$ with $\| \bar{u} - b \|_\infty = \| \bar{r} \|_\infty < \| r^* \|_\infty$. In this case, $r_i^* > \bar{r}_i$ if $r_i^* > 0$ and $e_i \in T^*$; $r_i^* < \bar{r}_i$ if $r_i^* < 0$ and $e_i \in T^*$. Thus, $u' = r^* - \bar{r} \in K$ conforms to r^* on T^* with $T^* \subseteq S(u')$. This contradicts $v \cdot u' = 0$, so u^* must satisfy (3.1).

Conversely, suppose that u^* satisfies (3.1). There is no $u' \in K$ such that $T^* \subseteq S(u')$ and u' conforms to r^* on T^*, or else for some $\lambda > 0$ we have $\| r^* - \lambda u' \|_\infty < \| r^* \|_\infty$. In Theorem 2.1, let $I_i = (0, \infty)$ if $r_i^* > 0$, $I_i = (-\infty, 0)$ if $r_i^* < 0$, $I_i = (-\infty, \infty)$ otherwise. Then, alternative (a) is false so there exists nonzero $v \in K^\perp$ such that $S(v) \subseteq T^*$ and v conforms to r^*. ■

The following is used in Section 5.

COROLLARY 3.1.1. *Let K be a subspace of R^n and let $b \in R^n$. Then*

$$\min_{u \in K} \| u - b \|_\infty = \max_{\text{nonzero } v \in K^\perp} v \cdot b / \| v \|_1 .$$

Proof. If $b \in K$ the statement is clearly true, since $v \cdot b = 0$ for all $v \in K^\perp$. Assume that $b \notin K$. For any $v \in K^\perp$, $| v \cdot b | = | v \cdot (u - b)| \leqslant \| v \|_1 \| u - b \|_\infty$ for all $u \in K$. Thus, $\| u - b \|_\infty \geqslant v \cdot b / \| v \|_1$ for all $u \in K$ and all nonzero $v \in K^\perp$. Let u^* be a solution of (3.1). By Theorem 3.1, there exists nonzero $v^* \in K^\perp$ such that $v^* \cdot (u^* - b) = \| u^* - b \|_\infty \| v^* \|_1 = -v^* \cdot b$. The equality of the Theorem is satisfied by u^* and $-v^*$. ■

The following gives necessary and sufficient conditions for a solution of (3.1) to be unique.

THEOREM 3.2. *Let K be a subspace of R^n and let $b \in R^n$, $b \notin K$. Let $u^* \in R^n$, $r^* = u^* - b$, $T^* = \{e_j : | r_j^* | = \| r^* \|_\infty\}$. Then, u^* is the unique solution* *of* (3.1) *if and only if there exists a spanning vector of* $K^\perp$, v, *such that* $S(v) = T^*$ *and* v *conforms to* r^*.

Proof. Suppose that two distinct vectors, u^* and u', satisfy (3.1). Let $r' = u' - b$. Then, $r^* - r' \in K$ conforms to r^* on T^*. Suppose that $v \in K^\perp$ satisfies $S(v) = T^*$ and conforms to r^*. Then, by orthogonality, $S(v) \cap S(r^* - r') = \varnothing$, so v is not a spanning vector of $K^\perp$.

Conversely, suppose that u^* is the unique solution of (3.1). There is no $\bar{u} \in K$ conforming to r^* on T^* else for some $\lambda > 0$, $\| r^* - \lambda\bar{u} \|_\infty \leqslant \| r^* \|_\infty$, contradicting (3.1) or uniqueness. In Theorem 2.1 (with K and $K^\perp$ interchanged), let $I_i = (0, \infty)$ if $r_i^* > 0$, $I_i = (-\infty, 0)$ if $r_i^* < 0$ and $I_i = \{0\}$ otherwise. Then, alternative (b) is false so (a) holds and there exists $v \in K^\perp$ such that $v_i \in I_i$, $i = 1,..., n$. That is, $S(v) = T^*$ and v conforms to r^*. Uniqueness of u^* implies that T^* intersects the support of every nonzero vector of K so v is a spanning vector of $K^\perp$. ∎

Note that "$S(v) = T^*$" can be replaced by "$S(v) \subsetneq T^*$" in the statement of the Theorem, by essentially the same proof.

Let $\mathrm{Dim}(K)$ denote the dimension of subspace K. We say that K *satisfies the Haar condition* if every nonzero $y \in K^\perp$ satisfies $|S(y)| > \mathrm{Dim}(K)$. (If either $K = R^n$ or $K = \{0\}$, then both K and $K^\perp$ satisfy the Haar condition, by application of the definition.) Assume that $K \neq R^n$, $K \neq \{0\}$. Let A be a matrix whose $m = \mathrm{Dim}(K)$ columns form a basis for K. Then, every linearly dependent set of rows of A forms a submatrix of rank m if and only if K satisfies the Haar condition. As previously noted, $y \in K^\perp$ is a spanning vector of $K^\perp$ if and only if the rows of A whose indices correspond to the indices of the nonzero components of y form a submatrix of rank m. Thus, K satisfies the Haar condition if and only if every nonzero $y \in K^\perp$ is a spanning vector of $K^\perp$. Fix nonzero $x' \in K$. If every nonzero $y \in K^\perp$ is a spanning vector of $K^\perp$, then $S(y) \cap S(x') \neq \varnothing$ for all nonzero $y \in K$ and hence, x' is a spanning vector of K. From the above, we conclude that K satisfies the Haar condition if and only if $K^\perp$ satisfies the Haar condition.

We now have a concise proof of a well-know result on uniqueness [5]:

THEOREM 3.3. *Let K be a subspace of R^n. Then,* (3.1) *has a unique solution for every* $b \in R^n$ *if and only if K satisfies the Haar condition.*

Proof. Suppose that K satisfies the Haar condition. Let u^* be a solution of (3.1). If $b \in K$ then, the unique solution of (3.1) is $u^* = b$. Assume that $b \notin K$. Since every nonzero $v \in K^\perp$ is a spanning vector of $K^\perp$, Theorems 3.1 and 3.2 imply that any solution of (3.1) is unique.

Conversely, suppose that K does not satisfy the Haar condition. Then, there exists nonzero $v \in K^\perp$ with $|S(v)| \leqslant \mathrm{Dim}(K)$. Choose $b_i = -1$ if $v_i > 0$, $b_i = +1$ if $v_i < 0$, $b_i = 0$ if $v_i = 0$. Since $v \cdot b < 0$, $b \notin K$. More-over, $u^* = 0$ solves (3.1) by Theorem (3.1), since $T^* = S(v)$ and $r^* = -b$. However, there is no spanning vector of $K^\perp$, w, satisfying $S(w) = T^*$, by cardinality of T^*. Thus, by Theorem 3.2, the solution $u^* = 0$ is not unique. ■

From the discussion of the Haar condition, we have the following corollary:

COROLLARY 3.3.1. *Let K_1 be a subspace of R^n. Then Eq.* (3.1), *with $K = K_1$, has a unique solution for all $b \in R^n$ if and only if Eq.* (3.1), *with with $K = K_1^\perp$, has a unique solution for all $b \in R^n$.*

When K fails to satisfy the Haar condition, there may exist *some* $b \notin K$ for which the solution of (3.1) is unique.

THEOREM 3.4. *Let K be a subspace of R^n. Then, there exists $b \in R^n$, $b \notin K$, such that* (3.1) *has a unique solution if and only if K has no vector with exactly one nonzero component.*

Proof. By Theorem 2.2, K has no vector with exactly one nonzero component if and only if $K^\perp$ possesses a spanning vector. Suppose that v is a spanning vector of $K^\perp$. Choose $b_i = -1$ if $v_i > 0$, $b_i = +1$ if $v_i < 0$, $b_i = 0$ if $v_i = 0$. By orthogonality, $b \notin K$. By Theorem 3.2, $u^* = 0$ is the unique solution of (3.1). Conversely, it follows from Theorem 3.2 that if $K^\perp$ has no spanning vector, then no solution of (3.1) with $b \notin K$ is unique. ■

4. Testing for Uniqueness

Now, we give a method for testing a solution of (3.1) for uniqueness. Let A be a matrix whose m columns form a basis for K. Assume that a solution of (3.1) u^*, is known and let T^* be as in Theorem 3.1. First, determine whether a nonzero vector of K exists with support disjoint from T^* by performing Gauss–Jordan elimination steps on the columns of the transpose of A whose indices correspond to T^*. If such operations reduce all these columns to zeros, then the solution u^* is not unique. Otherwise, determine whether the following equations and inequalities are consistent (e.g., by a linear programming algorithm):

$$\sum_{i=1}^{n} a_{ij} v_i = 0, \qquad j = 1,\dots, m$$

$$\begin{aligned} v_i &\geqslant 1, &&\text{if} \quad r_i^* > 0 \quad &&\text{and} \quad e_i \in T^* \\ v_i &\leqslant -1, &&\text{if} \quad r_i^* < 0 \quad &&\text{and} \quad e_i \in T^* \\ v_i &= 0, &&\text{if} \quad e_i \notin T^*. && \end{aligned}$$

By Theorem 3.2, the above is consistent if and only if u^* is unique.

5. A Special Case

Let $W = \{w_1, ..., w_t\}$ be a finite set of elements. Let E be a nonempty subset of $W \times W$ containing n elements. Let P be a given real-valued function on E. We wish to choose F, a real-valued function on W, so that

$$\max_{(w_i, w_j) \in E} |(F(w_j) - F(w_i)) - P[(w_i, w_j)]|$$

is minimized over all choices of F.

We represent the problem by a (directed) graph with n arcs and t vertices, where each arc corresponds to a member of $E = \{e_1, ..., e_n\}$. All graph theoretic terminology not explicitly defined here is identical to that used by Berge [2]. We assume that the graph is connected. (Otherwise, treat each connected component separately.)

Let G be a graph and let E be its set of n arcs and W its set of t vertices. A *tension* H, is a real-valued function on E having the property that there exists a real-valued *potential function* π, on W, such that $H[(w_i, w_j)] = \pi(w_j) - \pi(w_i)$. (If a tension is given and G is connected, a corresponding potential function is found by arbitrarily choosing $\pi(w_1)$.) Viewing a tension as a vector of R^n, it is not hard to show that the set of such *tension vectors* forms a subspace of R^n. Now, consider traversing an elementary cycle of G. We construct a vector of R^n, x, by setting $x_i = 0$ if e_i is not in the cycle, $x_i = +1$ if $e_i = (w_a, w_b)$ and is traversed from w_a to w_b, $x_i = -1$ if $e_i = (w_a, w_b)$ and is traversed from w_b to w_a. Any nonzero scalar multiple of a vector constructed in this way is called an *elementary circulation vector* of the graph. If $\{w_1, ..., w_c\}$ are the successive vertices of the cycle, then $\sum_{i: e_i \in E} x_i H(e_i) = \pi(w_2) - \pi(w_1) + \cdots + \pi(w_1) - \pi(w_c) = 0$. That is, any elementary circulation vector is orthogonal to the tension subspace. (In fact, it can be shown that these vectors span the orthogonal complement of the tension subspace.) Let $b = (b_1, ..., b_n)$ be defined by $b_k = P[(w_i, w_j)]$ for all $e_k = (w_i, w_j) \in E$. The problem posed above is solved by a solution to (3.1) with K the tension subspace of the associated graph.

We now give an algorithm for solution in this special case. In order to bound the number of steps, we assume that b is integral-valued. (If b is rational, multiply b by the least common multiple of the denominators of the components of b.)

Let $\tilde{b} = \max_i | b_i |$. Let $s \leqslant t$ be the maximum number of arcs in an elementary cycle of G. If v is an elementary circulation vector (of $K^{\perp}$), there exists $\lambda > 0$ such that $\lambda v = \bar{v}$ and such that $\bar{v}_i = 0, +1$ or -1. Since $b \cdot v/\| v \|_1 = b \cdot \bar{v}/\| \bar{v} \|_1$, the quantity $| b \cdot v |/\| v \|_1$, is a rational number of form p/q where $0 \leqslant p \leqslant \tilde{b}s$, $0 < q \leqslant s$.

We make use of an algorithm by Herz [1] for finding a "compatible tension." The algorithm (see Appendix I) finds $u \in K$ such that $k_i \leqslant u_i \leqslant l_i$

$\forall i \in J = \{1,..., n\}$ or, if none exists, finds an elementary circulation vector such that

$$\sum_{i:v_i>0} k_i v_i + \sum_{i:v_i<0} l_i v_i > 0.$$

In this application, the constants $\{k_i\}$ and $\{l_i\}$ are rational numbers of form p/q where $q \leqslant s$.

We begin with $k_i = l_i = b_i$ $\forall i \in J$. If a compatible tension is found, then $u^* = b$ is the solution of (3.1). If not, we obtain from the algorithm of Herz $v_1 \in K^\perp$ such that $v_1 \cdot b > 0$. Let $\delta_1 = v_1 \cdot b/\| v_1 \|_1$. By Corollary 3.1.1, $\min \| u - b \|_\infty \geqslant \delta_1$. We next seek $u' \in K$ such that for all $i \in J$, $-\delta_1 \leqslant u_i - b_i \leqslant \delta_1$. Let $k_i = b_i - \delta_1$, $l_i = b_i + \delta_1$ for all $i \in J$. If a compatible tension is found, then $u^* = u'$ is a solution of (3.1) with $\| u^* - b \|_\infty = \delta_1$. If not, we obtain $v_2 \in K^\perp$ such that $b \cdot v_2 - \delta_1 \| v_2 \|_1 > 0$. Let $\delta_2 = b \cdot v_2/\| v_2 \|_1 > \delta_1$. Then, $\min \| u - b \|_\infty \geqslant \delta_2$ by Corollary 3.1.1. We then seek u' such that $-\delta_2 \leqslant u_i' - b_i \leqslant \delta_2$ for all $i \in J$. We repeat until a compatible tension is found. Since the $\{\delta_j\}$ form a monotonic increasing sequence, none occurs more than once. Thus, the compatible tension algorithm is used no more than $\hat{b}s^2$ times.

In Appendix II, we prove the following:

THEOREM 5.1. *Let G be a connected graph and let K be the tension subspace of G. Let T be a subset of the arcs of G. Then, there exists v, a spanning vector of $K^\perp$, with $S(v) = T$ and $v_i \geqslant 0$ for $i = 1,..., n$ if and only if the graph obtained by deleting all arcs not in T is strongly connected.*

If $b \notin K$, the solution u^* is tested for uniqueness as follows: Delete all but the set of arcs for which $| u_i^* - b_i | = \| u^* - b \|_\infty$. Reverse the orientation (i.e., change $e_k = (w_a, w_b)$ to $e_k = (w_b, w_a)$) of any arc having $u_k^* - b_k < 0$. (This simply changes the sign of every nonzero kth component of a vector of $K^\perp$.) By Theorems 3.2 and 5.1, u^* is unique if and only if the modified graph is strongly connected. (An algorithm for testing for this property by labeling each vertex at most twice is given in [3].)

APPENDIX I

The following outlines an algorithm of Herz for finding a compatible tension [1, Chap. 5], using notation consistent with the preceding text. Let K be the tension subspace of a graph G. Let $\{e_1 ,..., e_n\}$ be the arcs of G. We seek $u \in K$ such that $u_i \in [k_i, l_i]$ for $i = 1,..., n$ where $\{k_i\}$ and $\{l_i\}$ are given integers. (If these bounds are rational, multiply all by an appropriate constant.) The *distance* of arbitrary u_i to the interval $[k_i, l_i]$ is defined by:

$$\begin{aligned} d_i(u_i) &= 0 \qquad \text{if} \quad u_i \in [k_i, l_i] \\ &= k_i - u_i \qquad \text{if} \quad u_i < k_i \\ &= u_i - l_i \qquad \text{if} \quad u_i > l_i . \end{aligned}$$

Starting with arbitrary integral u, we successively reduce the quantity $d(u) = \sum_{i=1}^{n} d_i(u_i)$. If $d(u) = 0$, the tension is compatible and the calculation terminates. If $d(u) > 0$, there exists an arc, say $e_1 = (b, a)$, such that $d_1(u_1) > 0$.

First, we suppose that $u_1 < k_1$, $d_1(u_1) = k_1 - u_1$. Vertex a is marked and marking of the remaining vertices is carried out as follows:

1. If x is already marked and y is not, mark y if $(x, y) = e_i$ is an arc with $u_i \leqslant k_i$.

2. If x is already marked and y is not, mark y if $(y, x) = e_i$ is an arc with $u_i \geqslant l_i$.

The marking terminates when no more vertices can be marked.

If vertex b is not marked, then $d(u)$ can be reduced. Define a potential function on the vertices taking value $+1$ if the vertex is marked, 0 if not. The corresponding tension $\bar{u}$, has the properties:

$$\bar{u}_i > 0 \qquad \text{if} \quad e_i = (b, a).$$

If $e_i \neq (b, a)$, then

$$u_i > 0 \Rightarrow u_i < l_i , \qquad u_i < 0 \Rightarrow u_i > k_i .$$

These properties imply that $d(u + \bar{u}) \leqslant d(u) - 1$. (In fact, the last step can often be improved upon by selecting the largest λ such that

$$\lambda \leqslant l_i - u_i , \qquad \text{for all } i \text{ with} \quad \bar{u}_i > 0,$$

$$\lambda \leqslant u_i - k_i , \qquad \text{for all } i \text{ with} \quad \bar{u}_i < 0.$$

Then, $d(u + \lambda\bar{u}) \leqslant d(u + \bar{u})$. Note that integrality of u implies integrality of λ and of the new tension $u + \lambda\bar{u}$.)

On the other hand, if vertex b *is* marked by the above procedure, there exists a chain from a to b in which each vertex has been marked using the previous vertex in accordance with the stated rules. In conjunction with (b, a), some subset of the arcs of this chain forms an elementary cycle. Let v be an elementary circulation vector whose support corresponds to this cycle and such that $v_1 = +1$. Then

$$0 = u \cdot v = \sum_{e_i : v_i > 0} u_i - \sum_{e_i : v_i < 0} u_i < \sum_{e_i : v_i > 0} k_i - \sum_{e_i : v_i < 0} l_i .$$

If this holds, then a vector satisfying $u_i' \in [k_i, l_i]$ for $i = 1,..., n$ would satisfy $u' \cdot v > 0$, so no such vector belongs to K.

The other case to consider is $e_1 = (b, a)$, $u_1 > l_1$, $d_1(u_1) = u_1 - l_1$. Essentially, the same procedure is followed. First, mark vertex a.

1. If x is marked and y is not, mark y if $(y, x) = e_i$ is an arc with $u_i \leqslant k_i$.

2. If x is marked and y is not, mark y if $(x, y) = e_i$ is an arc with $u_i \geqslant l_i$.

If b is not marked, obtain $\bar{u}$ from the potential function that takes value 0 on marked vertices, $+1$ on unmarked vertices. If b is marked, obtain v as above with $v_1 = -1$.

If the procedure is started with $u \equiv 0$, then initially,

$$d(u) \leqslant \sum_{i=1}^{n} \max(|k_i|, |l_i|).$$

Since each application of the marking procedure reduces $d(u)$ by at least one, the above expression bounds the number of such applications required to obtain a compatible tension or determine that none exists.

Appendix II

Proof of Theorem 5.1. Let G' be the strongly connected graph obtained by deleting from G all but the arcs of T. By a well-known property of strongly connected graphs [3], every arc belongs to an elementary circuit. Thus, there exists an elementary circulation vector with support contained in T, having no negative components, and such that a chosen arc belongs to its support. A sum of such vectors, one for each arc in T, is a non-negative vector of $K^{\perp}$ having support equal to T. Suppose that there exists $u \in K$ whose support is disjoint from T. Since T is incident with every vertex and since G' is connected, every potential function corresponding to u is constant on W and thus, $u \equiv 0$. Therefore, the vector constructed above is a spanning vector of $K^{\perp}$.

Conversely, suppose that v is a spanning vector of $K^{\perp}$, $S(v) = T$, $v_i \geqslant 0$ for $i = 1,..., n$. If G', the graph obtained by deleting the arcs not in T, is not strongly connected, then there exist $w_a \in W$ and $w_b \in W$ such that there is no path in G' from w_a to w_b. We define a potential function on G as follows: Set $\pi(w_a) = 1$. Set $\pi(w_i) = 1$ if there is a path from w_a to w_i in G'. Set $\pi(w_i) = 0$ if there is no path from w_a to w_i in G'. Since G is connected, the corresponding tension vector u, is nonzero. Also, $u_i \geqslant 0$ for $i = 1,\dots, n$. By orthogonality, $S(u)$ does not intersect $S(v)$, contradicting the spanning property of v. ∎

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